Property of Rational Functions Related to Band-Pass Transformation With Application to Symmetric Filters Design

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***Abstract*— This paper considers the condition when a rational function $F(s)$ may be represented as the function of the argument $s + (1/s)$. If this condition is satisfied then $F(s)$ is the ratio of recursive (symmetric) polynomials. This paper investigates the network properties of such rational functions and their realization. Then the symmetric polynomials are applied for synthesis of symmetric band-pass filters. Substituting $p = s + (1/s)$ into symmetric band-pass filter transfer function one obtains its low-pass generating filter. The slew rate and overshoot of generating filter step-response is closely connected with the step-response duration of symmetric band-pass filter. The choice of generating filter becomes an additional factor of symmetric band-pass filter design. As the generating filters the paper proposes using Lommel polynomial filters which have easy control of overshoot and slew rate. An example of six order symmetric band-pass filter is given.**

***Index Terms*— Bandpass transformation, Lommel polynomials, network theory, recursive polynomials, symmetric bandpass filters, symmetric polynomials.**

I. Introduction

RECENTLY, the interest towards classical network synthesis was on the rise again. This is connected with the development of RF circuits, in particular wide-band amplifiers [1]–[3] and their realization in new technologies. Power electronics [4], and the biomedical circuits providing the wavelet type response [5] also revived the interest towards classical time-domain analysis and synthesis of linear networks.

Band-pass filters occupy an important place in radio electronics from the early stages of its development, and many problems of their realization for specific requirements were investigated [6], [7]. The integrated realization of coils and, especially transformers, re-opened the interest to band-pass filters and amplifiers as the possibility to integrate passive and active band-pass filters using these elements [8]–[10]. It is also worthwhile to notice the attempts to adapt "bandpass-type" transformation for realization of active filters [11] and to generalize and extend the band-pass transformation for design of dual-band filters [12].

The goal of this paper is investigation of the band-pass filters with symmetric attenuation characteristics. Contrary to [13] where the symmetry is achieved by modification of traditional series and parallel resonators, i.e. at the final stage of realization, here the problem is concentrated on the properties of the band-pass transfer functions with symmetric amplitude-frequency responses (called for simplicity, in the paper title, as symmetric filters).

It worthwhile to mention that some microwave applications require band-pass filters with asymmetric characteristics. The methods and specifics of their microwave-based realization can be found in [[14], and references within]. An approach using fractional order transfer functions can be found in [15]. This work is using classical passive and active topologies but is limited by the filters based on second-order prototypes even though the higher order asymmetrical-slope band-pass filters may be realized by cascading.

The band-pass transformation is widely used in circuit theory and many properties of transfer functions subjected to this transformation are well known [16]–[19]. However, the theorem formulated below, which is closely related to this transformation, was practically inaccessible, the proof was too subtle and succinct [20], and required the knowledge of transcendental functions (see also Appendix C). In addition, that proof did not operate directly with recursive (or symmetric) polynomials; and, as a result, was not supported by any network applications. These deficiencies are corrected here. An elementary proof which has not been previously published, as far as the author is aware, is given and became the basis for network applications: it is used to construct the transfer function of symmetric band-pass filters. Special attention is paid to the step-response of these filters keeping in mind the applications described in [21].

The paper is organized as follows. Section II formulates the theorem. Section III gives the proof. Section IV describes some application of this theorem in the general network synthesis and filters. Section V reviews the known results for simple filters with symmetric amplitude-frequency responses, and introduces the idea of low-pass generating filter. The step-response of the generating filter becomes reflected in the step-response of band-pass filter. Section VI describes a higher order filter and the result of using Lommel polynomial in its generating filter. Section VII discusses the results. Appendix A provides the formulas useful for operations with symmetric polynomials and algebraic functions generating such polynomials. Appendix B provides the basic knowledge on Lommel polynomials. Appendix C provides the proof of the theorem using Chebyshev polynomials.

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II. Theorem

If $F(s)$ is a rational function of s, and $F(s) = F(1/s)$ for all s (except of singularities of $F(s)$), then $F(s)$ is the rational function of $s + (1/s)$.

III. Proof

Assume that

$$F(s) = \frac{s^k(a_0s^n + a_1s^{n-1} + \ldots + a_n)}{s^l(b_0s^m + b_1s^{m-1} + \ldots + b_m)} \tag{1}$$

where a_0, $b_0, \ldots a_n$, b_n are non-zero. Using the theorem condition one obtains that

$$\begin{aligned}&\frac{s^{2(l-k)+(m-n)}(a_ns^n + a_{n-1}s^{n-1} + \ldots + a_0)}{b_ms^m + b_{m-1}s^{m-1} + \ldots + b_0}\\ &\quad = \frac{a_0s^n + a_1s^{n-1} + \ldots + a_n}{b_0s^m + b_1s^{m-1} + \ldots + b_m}.\end{aligned} \tag{2}$$

The denominators of left and right parts have equal exponents, hence, the numerators should be of equal exponents as well. One can conclude that

$$m - n = 2(k - l) \tag{3}$$

and m and n are even or odd simultaneously. Then, using (2) and (3), one can write that

$$\begin{aligned}P_m(s) &= b_ms^m + b_{m-1}s^{m-1} + \ldots + b_0\\ &= b_0s^m + b_1s^{m-1} + \ldots + b_m\end{aligned} \tag{4}$$

and

$$\begin{aligned}P_n(s) &= a_ns^n + a_{n-1}s^{n-1} + \ldots + a_0\\ &= a_0s^n + a_1s^{n-1} + \ldots + a_n.\end{aligned} \tag{5}$$

Comparing coefficients in (4) and (5), one obtains that $a_0 = a_n$, $a_1 = a_{n-1}, \ldots, b_0 = b_m$, $b_1 = b_{m-1}, \ldots$, i.e., $P_m(s)$ and $P_n(s)$ are recursive (or symmetric) polynomials.

Let us now consider, for example, $P_n(s)$ and assume that n is even, i.e. $n = 2q$. One can rewrite (5) in this case as

$$P_{2q}(s) = s^q\left[\begin{array}{c}a_n\left(s^q + \dfrac{1}{s^q}\right) + a_{n-1}\left(s^{q-1} + \dfrac{1}{s^{q-1}}\right) + \\ \ldots + a_{q+1}\left(s + \dfrac{1}{s}\right) + a_q\end{array}\right]. \tag{6}$$

Now, one denotes $s + \frac{1}{s} = p$, and, using the binomial formula [22] one calculates

$$p^2 = s^2 + 2 + \frac{1}{s^2} \tag{7}$$

$$p^3 = s^3 + 3s + \frac{3}{s} + \frac{1}{s^3} \tag{8}$$

$$\begin{aligned}p^q &= s^q + qs^{q-2} + \frac{q(q-1)}{1\cdot 2}s^{q-4} + \ldots + \frac{q(q-1)}{1\cdot 2}\frac{1}{s^{q-4}}\\ &\quad + q\frac{1}{s^{q-2}} + \frac{1}{s^q} = \left(s^q + \frac{1}{s^q}\right)\\ &\quad + q\left(s^{q-2} + \frac{1}{s^{q-2}}\right) + \ldots\end{aligned} \tag{9}$$

From this set of equations, one finds successively

$$s + \frac{1}{s} = p \tag{10}$$

$$s^2 + \frac{1}{s^2} = p^2 - 2 \tag{11}$$

$$s^3 + \frac{1}{s^3} = p^3 - 3p \tag{12}$$

and so on. Hence

$$P_{2q}(s) = s^q\phi_q(p) \tag{13}$$

where $\phi_q(p)$ is a polynomial with the exponent of q.

Let us consider now the case of $n = 2q + 1$. Then

$$P_{2q+1}(s) = a_0s^{2q+1} + a_1s^{2q} + \ldots + a_{2q}s + a_{2q+1}. \tag{14}$$

This polynomial is symmetric, hence

$$a_0 = a_{2q+1}, a_1 = a_{2q}, a_2 = a_{2q-1}, \ldots \tag{15}$$

Then $P_{2q+1}(s)$ can be rewritten as

$$\begin{aligned}P_{2q+1}(s) &= a_0(s^{2q+1} + 1) + a_1(s^{2q} + s)\\ &\quad + a_2(s^{2q-1} + s^2) + \ldots + a_q(s^{q+1} + s^q)\\ &= a_0(s^{2q+1} + 1) + a_1s(s^{2q-1} + 1)\\ &\quad + a_2s^2(s^{2q-3} + 1) + \ldots + a_qs^q(s + 1).\end{aligned} \tag{16}$$

It is easy to verify that

$$\begin{aligned}s^{2m+1} + 1 &= (s + 1)(s^{2m} - s^{2m-1} + s^{2m-2}\\ &\quad - \ldots + s^2 - s + 1).\end{aligned} \tag{17}$$

Then the terms in (16) can be rewritten as

$$\begin{aligned}a_0(s^{2q+1} + 1) &= a_0(s + 1)(s^{2q} - s^{2q-1} + \ldots + s^2 - s + 1)\\ a_1s(s^{2q-1} + 1) &= a_1s(s + 1)(s^{2q-2} - s^{2q-3} + \ldots - s + 1)\\ &= a_1(s + 1)(s^{2q-1} - s^{2q-2} + \ldots - s^2 + s)\\ a_2s^2(s^{2q-3} + 1) &= a_2s^2(s + 1)(s^{2q-4} - \ldots + 1)\\ &= a_2(s + 1)(s^{2q-2} - \ldots + s^2)\\ a_qs^q(s + 1) &= a_q(s + 1)s^q.\end{aligned} \tag{18}$$

Summing these terms, one obtains that

$$P_{2q+1}(s) = (s + 1)Q(s) \tag{19}$$

where $Q(s)$ is the polynomial which is the sum of the following polynomials:

$$\begin{aligned}&a_0(s^{2q} - s^{2q-1} + s^{2q-2} - \ldots + s^2 - s + 1\\ &a_1(s^{2q-1} - s^{2q-2} + \ldots - s^2 + s)\\ &a_2(s^{2q-2} - \ldots + s^2)\\ &a_qs^q.\end{aligned} \tag{20}$$

But one can see that all polynomials in (20) are symmetric. Hence, their sum is also a symmetric polynomial of even exponent $n = 2q$, and one concludes that the summation result will be

$$P_{2q+1}(s) = (s + 1)s^q\psi_q(p) \tag{21}$$

where $\psi_q(p)$ is a polynomial with the exponent of q.

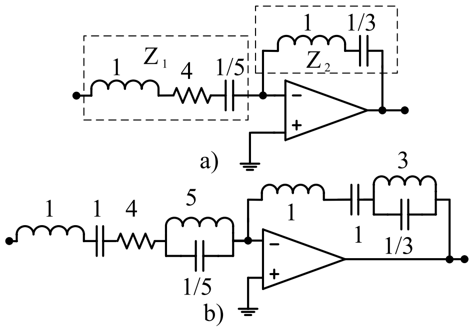

Fig. 1. Two-step transfer function realization.

Now one has two cases.

a) $m = 2r$ and $n = 2q$ are both even. Then, using (3) one obtains

$$F(s) = \frac{s^{k+q}\phi_q(p)}{s^{l+r}\psi_r(p)} = \frac{\phi_q(p)}{\psi_r(p)}s^{k-l+q-r} = \frac{\phi_q(p)}{\psi_r(p)}. \quad (22)$$

b) $m = 2r + 1$ and $n = 2q + 1$ are both odd. Then

$$F(s) = \frac{(s+1)s^{k+q}\phi_q(p)}{(s+1)s^{l+r}\psi_r(p)} = \frac{\phi_q(p)}{\psi_r(p)}s^{k-l+q-r} = \frac{\phi_q(p)}{\psi_r(p)}. \quad (23)$$

The theorem is proven.

IV. Application to Network Synthesis

Let us consider the function

$$F(s) = \frac{s^4 + 5s^2 + 1}{s^4 + 4s^3 + 7s^2 + 4s + 1}. \quad (24)$$

It is easy to verify that this function satisfies the theorem condition.

Let be required that (24) is realized as a network transfer function and the sign obtained in realization is not important. Dividing the numerator and denominator of (24) by s^2 and using the results (10) – (12) one can represent (24) as

$$F(p) = \frac{p^2 + 3}{p^2 + 4p + 5} \quad (25)$$

where $p = s + (1/s)$. Now, one can consider realization of (25) as a transfer function for the variable p. If one decides to realize (25) using an ideal operational amplifier, one can introduce

$$Z_1(p) = \frac{p^2 + 4p + 5}{p} = p + 4 + \frac{5}{p} \quad (26)$$

and

$$Z_2(p) = \frac{p^2 + 3}{p} = p + \frac{3}{p}. \quad (27)$$

The realization using $Z_1(p)$, $Z_2(p)$ and, say, an ideal operational amplifier is shown in Fig. 1 a.

Finally, considering that $p = s+(1/s)$ and $1/p = s/(s^2+1)$ each inductor in the network of Fig. 1a is substituted by the series connection of inductor and capacitor, and each capacitor is substituted by the parallel connection of inductor and capacitor. This final realization is shown in Fig. 1b. Hence, the transition to the p-variable, in case if the transfer function satisfies the theorem condition, may be considered as an intermediate step simplifying the realization. The reader should not worry about physical implementation of this example; in network theory we are dealing with ideal elements.

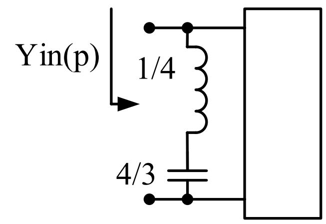

Fig. 2. Positive realness verification.

The transition to the p-variable may be useful in case of one-port realization as well. If it be required that (24) is realized as the input impedance, and the realization should be passive, the answer will be negative. The transformation $p = s+(1/s)$ does not change the property of positive realness of the transformed function [18]. Then, if

$$Z_{in}(p) = \frac{p^2 + 3}{p^2 + 4p + 5}. \quad (28)$$

then (Fig. 2)

$$Y_{in}(p) = \frac{p^2 + 4p + 5}{p^2 + 3} = \frac{4p}{p^2 + 3} + \frac{p^2 + 5}{p^2 + 3}. \quad (29)$$

But it is easy to see that for $p = j\Omega$ the second term of (29) is the real part of $Y_{in}(j\Omega)$, i.e.,

$$ReY_{in}(j\Omega) = \frac{\Omega^2 - 5}{\Omega^2 - 3} \quad (30)$$

and this real part is negative for $3 < \Omega^2 < 5$. Hence, the second term of (29) can not be realized as an input admittance of a passive one-port, and (28) (and, hence (24)) is not realizable as a passive one-port either.

V. Simple Band-Pass Filters With Symmetric Amplitude-Frequency Response

Let us consider a simple example: symmetric band-pass filters with fourth order polynomial in the transfer function denominator

$$T(s) = \frac{Ks^2}{a_4s^4 + a_1s^3 + a_2s^2 + a_1s + a_0}. \quad (31)$$

One well-known example of symmetric 4^{th} order polynomial is the fourth order Butterworth polynomial. The band-pass filter using this polynomial has the transfers function

$$T_1(s) = \frac{K_1s^2}{s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1}. \quad (32)$$

Dividing the numerator and denominator of (32) by s^2 and using the results (10) –(12) one can represent (32) as

$$T_1\left(s + \frac{1}{s}\right) = G_1(p) = \frac{K_1}{p^2 + 2.6131p + 1.4142}. \quad (33)$$

The function $G_1(p)$ (33) will be called "generating filter" for the transfer function $T_1(s)$. The substitution $p = s + (1/s)$

in (33) will return us back to (32). From the other side, $G_1(p)$ is a low-pass filter transfer function (of variable p), and the step response parameters of $G_1(p)$, (especially slew-rate and overshoot) define the step-response duration for $T_1(s)$.

If one considers

$$G_2(p) = \frac{K_2}{p^2 + 1.4142p + 1} \tag{34}$$

and applies the transformation $p = s + (1/s)$ to this generating filter one obtains

$$T_2(s) = \frac{K_2 s^2}{s^4 + 1.4142s^3 + 3s^2 + 1.4142s + 1}. \tag{35}$$

This band-pass filter is known as "double-tuned inter-stage coupling network". The amplitude-frequency response of this filter is well investigated [19], [23], and we used it here to demonstrate that application of band-pass transformation to a symmetric polynomial results in a symmetric polynomial of higher order.

We can also obtain a symmetric forth order band-pass filter if we multiply the transfer functions of two symmetric band-pass filters of second order. For example, one obtains

$$T_3(s) = \frac{K_3 s^2}{(s^2 + 1.4142s + 1)^2} = \frac{K_3 s^2}{s^4 + 2.8284s^3 + 4s^2 + 2.8284s + 1}. \tag{36}$$

This band-pass filter is known as "network with identical cascaded resonators" [23], [24]. The generating filter for $T_3(s)$ is given by

$$G_3(p) = \frac{K_3}{p^2 + 2.8284p + 2}. \tag{37}$$

Let us start to compare these three filters. They are called here "Butterworth-derived" just because in this part we operated with Butterworth polynomials; the general approach is considered in the next part. Fig. 3 shows their amplitude and phase frequency responses. Fig. 3a shows that $T_1(s)$ is the most narrow-banded filter. $T_2(s)$ and $T_3(s)$ are more wide-banded filters (this is why these configurations are considered in wide-band selective amplifiers), but in a different manner: at the point of $\omega = 1$ the phase characteristic of $T_2(s)$ is steeper than that of $T_1(s)$ and the phase characteristic of $T_3(s)$ is less steep than that of $T_1(s)$.

The steepness of phase frequency characteristic at the point $\omega = 1$ may be connected with the duration of the filter step response. The step responses of $T_1(s)$, $T_2(s)$ and $T_3(s)$ are shown in Fig. 4. Indeed, the step transient response of $T_2(s)$ is the longest one.

Additional design information is obtained considering the step responses of generating filters $G_1(p)$, $G_2(p)$ and $G_3(p)$. These are shown in Fig. 5. Comparing the graphs of Fig. 4 and Fig. 5 one can conclude that the generating function having the highest slew rate at the beginning of the response, and, at the same time, not having any overshoot will provide the shortest duration for the step response of the corresponding bandpass filter.

The realization procedure of the bandpass filters considered in this part is common to all of them. The first step is realization of the corresponding generation filter. If, for example, one

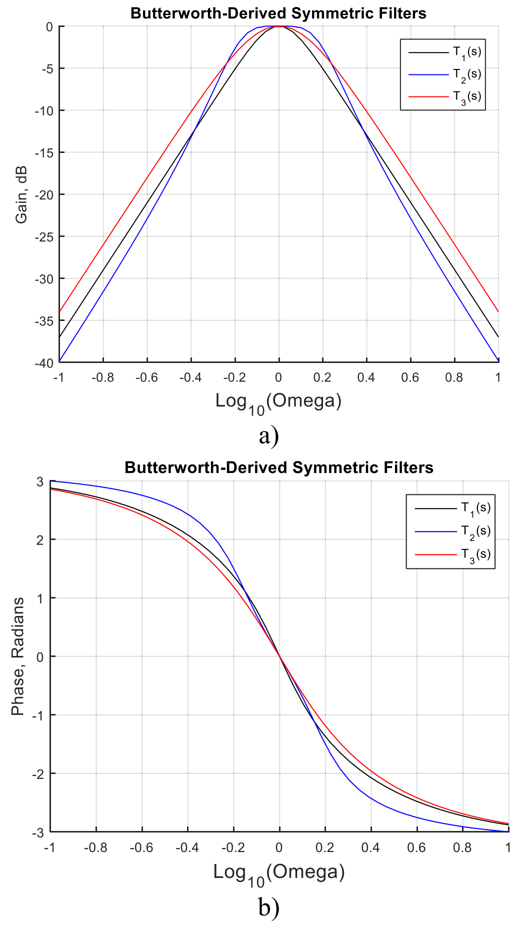

Fig. 3. Frequency characteristics of the derived filters.

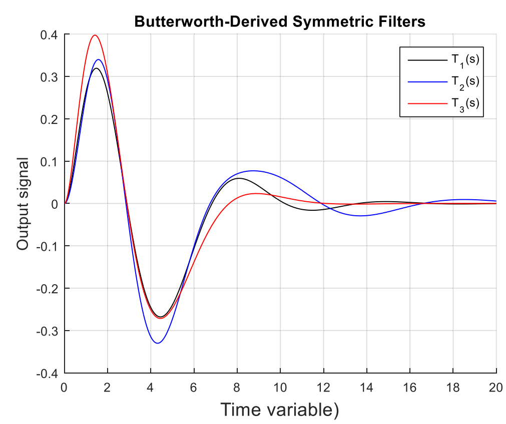

Fig. 4. Comparison of filters step responses.

considers realization of $G_1(p)$ with $K_1 = 1$ (in the previous consideration this gain constant was chosen different for easy comparison with the characteristics of other filters) then a network with this transfer function can be easily found. It is shown in Fig. 6a. In this network $R_2 = 1$, $R_1 = 0.4142$ and the inductor and capacitor can be found from the system of two equations

$$\begin{cases} LC = 1 \\ L + C = 2.6131. \end{cases} \tag{38}$$

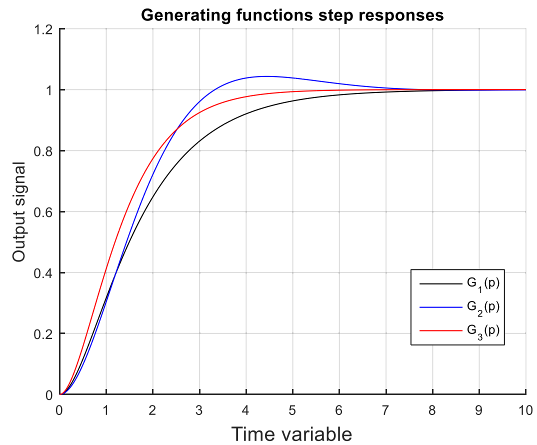

Fig. 5. Step responses of generating functions.

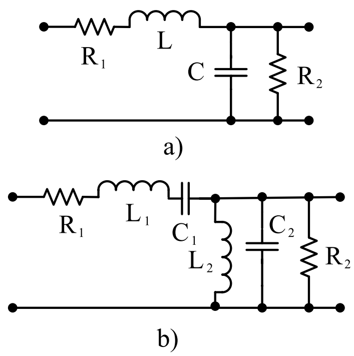

Fig. 6. Realization of Butterworth-derived filters.

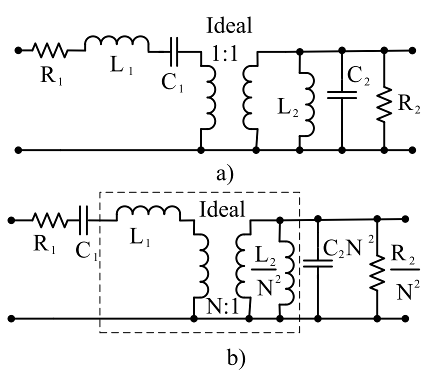

Fig. 7. Idealized models of electromagnetic energy transducers.

These equations have two solutions: $L = 2.1474$, $C = 0.4657$, and $L = 0.4657$, $C = 2.1474$.

Finally, the inductor in the network of Fig. 6a is substituted by series connection of the inductor and capacitor, and the capacitor is substituted by parallel connection of the inductor and capacitor. This final realization is shown in Fig. 6b. Notice that this realization includes the source impedance and the load (even though they are not equal; using equal loads at both ends is possible but it would reduce the filter gain).

The developed approach not only unite the previously known symmetric bandpass filters but may be also connected with design of electromagnetic energy transducers [25], [26] recently became important in energy harvesters.

If one introduces an ideal transformer with 1:1 turn ratio [as shown in Fig. 7a] the transfer function of this network will be the same as that of Fig. 6b. One can do a step further and use the transformer with N:1 turn ratio [Fig. 7b]. This step, of course, requires the change of impedance level of elements in the secondary.

But the network shown in Fig. 7b is one of equivalent networks of the electromagnetic energy transducers with strong coupling. Hence, the proposed approach allows to introduce the approximation problem at the early stages of transducer design: $T_1(s)$ should be used when the energy is concentrated in the narrow band, $T_2(s)$ provides larger bandwidth (and longer transient response, and $T_3(s)$ is an intermediate compromise solution. Then, Fig. 6a elements are obtained for $R_2 = 1$, and $R_1 = 0.4142$. Using other resistors (this is the advantage of the synthesis!) and varying N may help to better match the network model with the transducer circuit using real elements.

VI. Higher Order Bandpass Filters with Symmetric Amplitude-Frequency Response

As one can see, the proposed theorem allows one to simplify the synthesis of bandpass polynomial filters at the condition that their denominators are recursive (symmetric) polynomials: introducing the inverse transformation of variables reduces by two times the order of the denominator polynomial. The Butterworth polynomials are not the only ones which have this property.

The simplest way to obtain the recursive polynomials is to consider the polynomial including also terms with negative exponents

$$P_n(s) = \frac{1}{s^n} Q_n(s) + s^n Q_n\left(\frac{1}{s}\right) \tag{39}$$

where $Q_n(s)$ is an arbitrary polynomial of the degree n. Indeed, one can see that $P_n(s) = P_n(1/s)$, and, hence, $P_n(s)$ is a recursive polynomial

$$P_n(s) = \begin{bmatrix} \left(s^n + \frac{1}{s^n}\right) + a_{n-1}\left(s^{n-1} + \frac{1}{s^{n-1}}\right) + \\ \ldots + a_1\left(s + \frac{1}{s}\right) + a_0 \end{bmatrix}. \tag{40}$$

Then one can see that

$$T(s) = \frac{K}{P_n(s)} \tag{41}$$

may be immediately used as the transfer function of a symmetric bandpass filter at the condition that the polynomial $Q_n(s)$ provides physical realization of (41).

But in addition to physical realizability there are requirements to the filter bandwidth and the transient response duration. If we use the results (10) – (12) (the missing terms can be found or developing the sequence (10) – (12) further or using the formulas given in Appendix A) then

$$H_n(s) = \frac{a_0}{P_n(p)} \tag{42}$$

is a generating filter for the transfer function (41). In accordance with observations made in the previous part the step-transient response of this generating function should be fast yet without overshoot (or with a very small overshoot). $H_n(s)$ is a

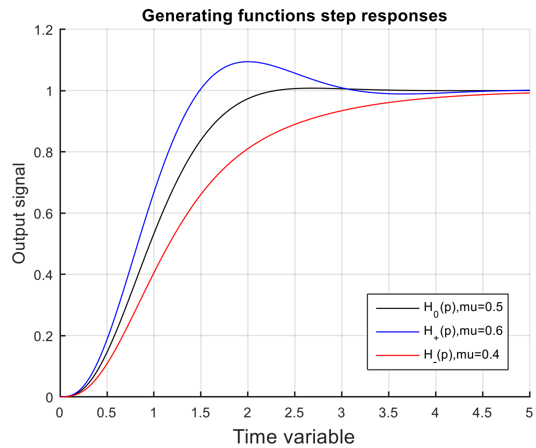

Fig. 8. Step-transient responses for third order Lommel polynomial filters.

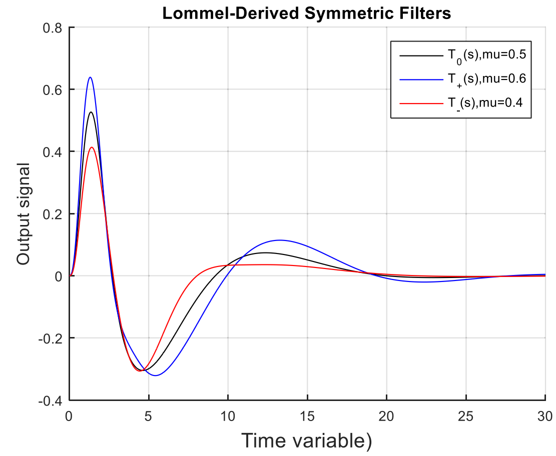

Fig. 9. Step transient responses of symmetric filters.

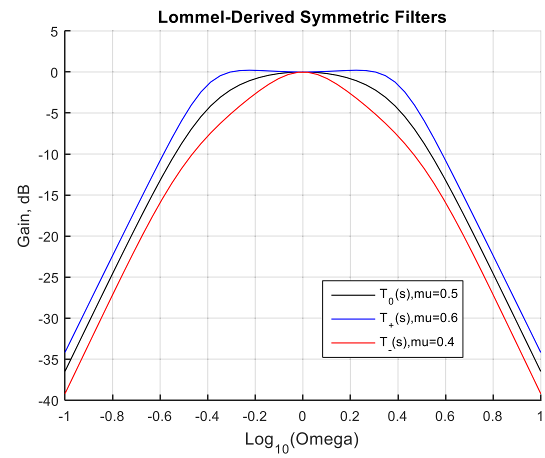

Fig. 10. Amplitude-frequency responses of the bandpass filters.

low-pass filter transfer function. The problem of step-transient response without overshoot in wideband amplifiers/filters is well known [28], [29], see also bibliography in both sources].

We propose to use for the considered task, as the denominators of generating filters, Lommel polynomials [30] (see also Appendix B): using Lommel polynomial provides an easy control of overshoot in the step response. For example, the second order generating filter with second order Lommel polynomial in the denominator will be

$$H_{L2}(p) = \frac{4\mu(\mu+1)}{p^2 + 2\nu p + 4\mu(\mu+1)}. \tag{43}$$

One can see that it is easy to find the values of μ and ν which are used in (33), (34) and (37).

As it was shown in [30], the Lommel polynomials are centered around Bessel polynomials, and for $\mu = 1/2$ and $\nu = 3/2$ both groups of polynomials are coinciding. But it is well known [23] that Bessel polynomial filters are having the step response without overshoot. Hence, Bessel polynomial filters may be considered as a good initial choice for generating filters. Using different values of μ and ν one can modify the generating filter step transient response, and hence, find the compromise between the bandpass and transient duration for the bandpass filter derived from this generating filter.

As an example, consider the synthesis of bandpass filter with the six order symmetric polynomial in the denominator. In accordance with the proposed approach, one starts design considering the third-order generating filter

$$H_{L3}(p) = \frac{a_0}{p^3 + a_2 p^2 + a_1 p + a_0}. \tag{44}$$

Here, $a_0 = 8\mu(\mu+1)(\mu+2)$, $a_1 = 4\nu(\nu+1)$. and $a_2 = 4(\mu+1)$.

Fig. 8 shows the step transient responses for $\nu = 3/2$ and three different values of μ (the corresponding subscript $L3 = 0, +, -$). One can see that increasing μ one obtains step transient responses with higher speed but with increasing overshoot.

The transfer functions of the symmetric filters are obtained substituting $p = s + (1/s)$ into (44). One obtains

$$T_i(s) = \frac{a_0 s^3}{\begin{bmatrix} s^6 + a_2 s^5 + (3+a_1)s^4 + (2a_2+a_0)s^3 \\ +(3+a_1)s^2 + a_2 s + 1 \end{bmatrix}} \tag{45}$$

where $i = 0, +, -$. The step transient responses of the filters $T_i(s)$ are shown in Fig. 9. One can see, indeed, that overshoot in the step response of generating filter results in a longer transient response of the filter itself.

It is possible to show that the slew rates of the generating filter and the symmetric filter for small values of time variable are coinciding. This results in a faster initial response (with a larger initial amplitude, of course) of the filter derived from the generation filter with faster step response.

Fig. 10 demonstrates amplitude frequency responses of the obtained filters. One can see that the bandwidth increase is realized in the similar way as in Fig. 3(a), by making higher the "shoulders" of the response.

The realization of the chosen filter can be done in two steps: first, we realize a generating low-pass filter, and then, using low-pass to bandpass transformation do substitution of reactance elements be resonant circuits. If, for example, we choose $\mu = 0.5$ and $\nu = 3/2$, the generating filter transfer

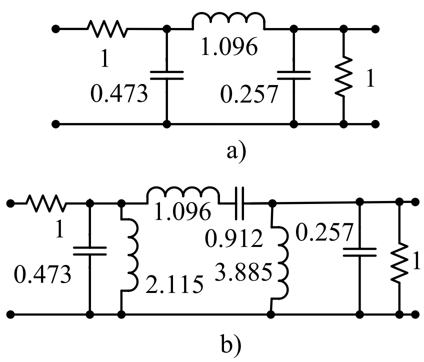

Fig. 11. Realization of 6th-order symmetric filter.

function becomes

$$H_0(p) = \frac{15}{p^3 + 6p^2 + 15p + 15}. \tag{46}$$

It is easy to verify that the network shown in Fig. 11a realizes the transfer function

$$H_0(p) = \frac{7.5}{p^3 + 6p^2 + 15p + 15} \tag{47}$$

Doing the substitution of reactance elements in accordance with low-pass to bandpass transformation one arrives to the network shown in Fig. 11b which realizes the transfer function

$$T_0(s) = \frac{7.5s^3}{s^6 + 6s^5 + 18s^4 + 27s^3 + 18s^2 + 6s + 1}. \tag{48}$$

Realization of higher order low-pass generating filters is a difficult task. A particular case is known as the design of triple resonance inter-stage networks [31], [32]. It is a special not well-investigated problem and is not considered here.

VII. Discussion and Conclusion

The investigation of condition when a rational function $F(s)$ becomes the function of the argument $p = s + (1/s)$ and the proof of corresponding theorem resulted in consideration of transfer functions which are ratios of recursive (symmetric) polynomials and symmetric bandpass filters in particular.

Substituting $p = s + (1/s)$ in the transfer functions of symmetric bandpass filters gives their low-pass generating filters. The step-response of generating filter is influencing the step-response of bandpass filter and may be used as an additional factor for the choice of the bandpass transfer function. The known examples of symmetric bandpass filters are different in their generating filter functions.

Realization of symmetric bandpass filter should start from realization of generating filter. Different solutions which normally obtained at this stage allow to choose more favorable result for transition to electronic circuit.

The theorem which establishes the condition when any analytic function of s may be represented as the function of $p = s + (1/s)$ [20] was not considered here; we did not find any interesting network applications; perhaps an interested reader will fill this gap.

In accordance with the long-time request of [33], a partial investigation of the theorem implications on the network synthesis and on analog filter design was provided. The technique proposed as the result of this theorem proof combines time domain approximation (the generating filter is chosen on the basis of time-domain response) with frequency-domain low-pass to bandpass transformation.

A possible extension of the theorem on design of digital filters [34] and, possibly, two-dimensional digital filters may be a matter of the future work.

Appendix A

Calculating $s^k + (1/s^k)$ does not require the sequence of steps described in (7) – (12). Two formulas are useful [35]. The first one is

$$\frac{1}{k}\left(s^k + \frac{1}{s^k}\right) = \frac{1}{k}p^k - \frac{(k-2)!}{1!(k-2)!}p^{k-2} + \frac{(k-3)!}{2!(k-4)!}p^{k-4} - \frac{(k-4)!}{3!(k-6)!}p^{k-6} + \ldots \tag{A.1}$$

The terms on the right side of (A.1) can be obtained by the common procedure. The general expression for the term is

$$\frac{(-1)^m(k-m-1)!}{m!(k-2m)!}p^{k-2m} \tag{A.2}$$

where $m = 0, 1, 2, \ldots$ up to the maximum value of m when the exponent $k - 2m$ is still positive (i.e. m is an integer and $m \le \frac{k}{2}$). Hence, (A.1) can be also written as the sum

$$\frac{1}{k}\left(s^k + \frac{1}{s^k}\right) = \sum_{m=0}^{p} \frac{(-1)^m(k-m-1)!}{m!(k-2m)!}p^{k-2m} \tag{A.3}$$

where m is a maximal integer for which $m \le (k/2)$.

Appendix B

The Lommel polynomials [30, 36, 37], $P_{nLo}(s, \mu, \nu)$, of complex variable s and two real value parameters μ and ν, are given by the following expressions:

$$P_{1Lo}(s, \mu, \nu) = 2\mu + s \tag{B.1}$$

$$P_{2Lo}(s, \mu, \nu) = 4\mu(\mu+1) + 2\nu s + s^2 \tag{B.2}$$

$$P_{3Lo}(s, \mu, \nu) = 8\mu(\mu+1)(\mu+2) + 4\nu(\nu+1)s + 4(\mu+1)s^2 + s^3 \tag{B.3}$$

$$P_{4Lo}(s, \mu, \nu) = 16\mu(\mu+1)(\mu+2)(\mu+3) + 8\nu(\nu+1)(\nu+2)s + 12(\mu+1)(\mu+2)s^2 + 4(\nu+1)s^3 + s^4. \tag{B.4}$$

If in (B.1) – (B.4) one takes $\mu = 1/2$ and $\nu = 3/2$ then one finds that

$$P_{nLo}(s, 1/2, 3/2) = P_{nBe}(s) \tag{B.5}$$

i.e. Lommel polynomials turn into Bessel polynomials, $P_{nBe}(s)$, in the form as they are used in filter design [36]. Indeed, doing calculations one obtains

$$P_{1Lo}(s, 1/2, 3/2) = 1 + s = P_{1Be}(s) \tag{B.6}$$

$$P_{2Lo}(s, 1/2, 3/2) = 3 + 3s + s^2 = P_{2Be}(s) \tag{B.7}$$

$$P_{3Lo}(s, 1/2, 3/2) = 15 + 15s + 6s^2 + s^3 = P_{3Be}(s) \tag{B.8}$$

$$P_{4Lo}(s, 1/2, 3/2) = 105 + 105s + 45s^2 + 10s^3 + s^4 = P_{4Be}(s). \tag{B.9}$$

APPENDIX C

A proof requiring the knowledge of transcendent functions may be represented as following [20].

Let $F(s) = M(s)/N(s)$, where $M(s)$ and $N(s)$ are polynomials, and $F(s) = F(1/s)$. Then

$$F(s) = \frac{M(s)}{N(s)} = \frac{M(1/s)}{N(1/s)} = \frac{M(s) + M(1/s)}{N(s) + N(1/s)}. \tag{C.1}$$

Write $p = (1/2)[s + (1/s)]$, and $s = e^{j\theta}$ (where θ may be complex). Then $p = \cos\theta$ and

$$s^k + (1/s^k) = 2\cos k\theta = 2T_k(p), k = 0, 1, 2, \ldots \tag{C.2}$$

where $T_k(p)$ is a Chebyshev polynomial [37]. It follows at once that $M(s) + M(1/s)$ and $N(s) + N(1/s)$ are both polynomials in p and hence that $F(s)$ in rational in p.

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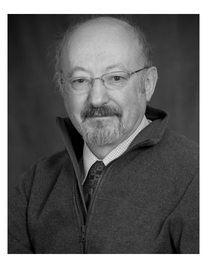

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